MSE Hand in 7 - Solution

Assignment 1

Consider the following system of equations

$$\begin{aligned} 2x_2 + 3x_3 + 4x_4 &= 1 \\ x_1 - 3x_2 + 4x_3 + 5x_4 &= 2 \\ -3x_1 + 10x_2 - 6x_3 - 7x_4 &= -4 \end{aligned}$$

a. Set up the coefficient matrix A of the above system.

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 1 & -3 & 4 & 5 \\ -3 & 10 & -6 & -7 \end{bmatrix}$$

b. Determine the steps.

Step 1: $r_1 \leftrightarrow r_2$

Step 2: $r_3 \rightarrow r_3 + 3r_1$

Step 3: $r_2 \leftrightarrow r_3$

Step 4: $r_3 \rightarrow r_3 - 2r_2$

Step 5: $r_3 \rightarrow \frac{-1}{9} r_3$

c. General Solution

x_1, x_2, x_3 basic (pivot columns) and x_4 free (non-pivot columns)

$$\begin{aligned} x_1 - \frac{1}{3}x_4 &= \frac{2}{3} \\ x_2 &= 0 \\ x_3 + \frac{4}{3}x_4 &= \frac{1}{3} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} x_1 &= \frac{2}{3} + \frac{1}{3}x_4 \\ x_2 &= 0 \\ x_3 &= \frac{1}{3} - \frac{4}{3}x_4 \\ x_4 &= x_4 \end{aligned}$$

d. Parametric Solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ \frac{1}{3} \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \frac{1}{3} \\ 0 \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

Assignment 2

You are given the following system of equations

$$\begin{cases} x_1 - 2x_2 - 9x_4 = 8 \\ -x_2 + 4x_3 + 3x_4 = -1 \\ -2x_1 + x_2 + x_3 + 5x_4 = -8 \end{cases}$$

a. Vector form of the system of equations.

$$x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} -9 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ -8 \end{bmatrix}$$

b. Matrix Equation

$$\begin{bmatrix} 1 & -2 & 0 & -9 \\ 0 & -1 & 4 & 3 \\ -2 & 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ -8 \end{bmatrix}$$

c. True or failures

- **The system is consistent (TRUE)**
  Since in the RREF there is no row of the form $[0\,0\,\cdots\,|\,b]$ with $b \neq 0$.
- **There is a unique solution (FALSE)**
  Because x_4 is a free variable since there is no pivot in the fourth column.
- **Span** $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} a \\ 3 \\ 5 \end{bmatrix} \right\} = \mathbb{R}^3$ **(TRUE)**
  The columns in A span $\mathbb{R}^3$ because there is a pivot in every row of rref(A).
- $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -9 \\ 3 \\ 5 \end{bmatrix} \right\}$ **are linearly independent (FALSE)**
  Because there are more vectors than entries in each vector. From the RREF, it can be deduced that...

$$\begin{bmatrix} -9 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + 5 \cdot \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$$

- **The solution set of the system of equations is a line in $\mathbb{R}^3$ (FALSE)**
  Since the solutions will have five entries:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^Y.$$

- $$\begin{bmatrix} -9 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + 5 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} \quad \textbf{(TRUE)}$$

  This can be deduced from the RREF.

Assignment 3

For each of the sets of vectors below determine whether the set is linearly dependent or linearly independent. Notice: you do not need to do a lot of calculations.

A. Linearly dependent since there are more vectors than entries in each vector.

B. Linearly dependent since it contains the zero vector.

C. Linearly dependent since $\begin{bmatrix} -4 \\ 4 \end{bmatrix} = -4 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

D. Linearly dependent since $\begin{bmatrix} -6 \\ 9 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$

E. Linearly independent since the vectors are not scalar multiples of each other, i.e. $\nexists k \in \mathbb{R}$ such that $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Assignment 4

a. You are given the matrix $A = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} -8 \\ 16 \\ 4 \end{bmatrix}$. Write $\mathbf{b}$ as a linear combination of the columns from A

Reduce the augmented matrix

$$\begin{bmatrix} 3 & -2 & 0 & -8 \\ 1 & 2 & 4 & 16 \\ 0 & 0 & 1 & 4 \end{bmatrix} r_2 \leftrightarrow r_1 \begin{bmatrix} 1 & 2 & 4 & 16 \\ 3 & -2 & 0 & -8 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$r_2 \to r_2 - 3r_1 \begin{bmatrix} 1 & 2 & 4 & 16 \\ 0 & -8 & -12 & -56 \\ 0 & 0 & 1 & 4 \end{bmatrix} r_2 \to \frac{-1}{8} r_2 \begin{bmatrix} 1 & 2 & 4 & 16 \\ 0 & 1 & \frac{3}{2} & 7 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$r_2 \to r_2 - \frac{3}{2} r_3 \begin{bmatrix} 1 & 2 & 4 & 16 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} r_1 \to r_1 - 4r_3 \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$r_1 \to r_1 - 2r_2 \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

so, $\mathbf{b} = -2\mathbf{a}_1 + 1\mathbf{a}_2 + 4\mathbf{a}_3$

b. You are given the matrix $A = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 2 & 4 \\ 4 & -4 & 4 \end{bmatrix}$. Write $\mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix}$ as a linear combination of the columns of A.

The method is similar to the previous question. The RREF of the augmented matrix is $\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Setting $x_3 = -2$, we get: $\mathbf{b} = 8\mathbf{a}_1 + 4\mathbf{a}_2 - 2\mathbf{a}_3$

c. You are given the matrix $A = \begin{bmatrix} 0 & 2 & 4 \\ 1 & -2 & -5 \\ 0 & 3 & 6 \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} h \\ 1 \\ 9 \end{bmatrix}$. Find the value of h such that $\mathbf{b}$ is a linear combination of the columns of A.

The method is similar to the previous questions. The EF of the augmented matrix is $\begin{bmatrix} 1 & -2 & -5 & 1 \\ 0 & 0 & 0 & h-6 \\ 0 & 1 & 2 & 3 \end{bmatrix}$.

That means we need $h - 6 = 0 \Leftrightarrow h = 6$.

Assignment 5

a. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ can be translated to $x_1 + x_2 = 0$ where x_2 is a free variable. The general solution is $x_1 = -x_2$ and the parametric solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. This is a line through the origin in $\mathbb{R}^2$ with the direction vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ - or the line that goes through the origin and has a slope of -1.

b. $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ can be translated to $x_1 + x_2 = 2$ where x_2 is a free variable. The parametric solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. This is a line through (2, 0) in $\mathbb{R}^2$ with the direction vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ - or the line that goes through (2, 0) and has a slope of -1 (it intercepts the vertical axis at 2).

c. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ can be translated to $x_1 = 2$ and $x_2 = 3$. This is just a point in $\mathbb{R}^2$ at (2, 3).

d. $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ can be translated to $x_1 - x_2 = 0$ where x_2 is a free variable. The general solution is $x_1 = x_2$ and the parametric solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This is a line through the origin in $\mathbb{R}^2$ with the direction vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ - or the line that goes through the origin and has a slope of 1.

Assignment 6

You are given a matrix $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -2 & 0 & -2 & -2 \\ 1 & 2 & 3 & -1 \end{bmatrix}$ and a vector $\mathbf{b} = \begin{bmatrix} -2 \\ 4 \\ 8 \end{bmatrix}$.

a. Write the general solution of $A\mathbf{x} = \mathbf{b}$ in parametric vector form.

We reduce to reduced echelon form:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & -2 \\ -2 & 0 & -2 & -2 & 4 \\ 1 & 2 & 3 & -1 & 8 \end{bmatrix} \xrightarrow[r_3 \to 2r_3 - r_1]{r_2 \to r_2 + 2r_1} \begin{bmatrix} 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & -2 & 10 \end{bmatrix}$$

$$r_3 \leftrightarrow r_2 \begin{bmatrix} 1 & 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & -2 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} r_2 \to \frac{1}{2} r_2 \begin{bmatrix} 1 & 0 & 1 & 1 & -2 \\ 0 & 1 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:

$$\begin{aligned} x_1 + x_3 + x_4 &= -2 \\ x_2 + x_3 - x_4 &= 5 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} x_1 &= -2 - x_3 - x_4 \\ x_2 &= 5 - x_3 + x_4 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

b. Write the general solution to the homogenous equation $A\mathbf{x} = \mathbf{0}$ in parametric vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

c. $\begin{bmatrix} -2 \\ 5 \\ 0 \\ 0 \end{bmatrix}$ is a solution to the inhomogenous equation $A\mathbf{x} = \mathbf{b}$ from part (a). (TRUE) since you can choose $x_3 = 0$ and $x_4 = 0$ and then $x_1 = -2$ and $x_2 = 5$.

$\begin{bmatrix} -2 \\ 5 \\ 0 \\ 0 \end{bmatrix}$ is a solution to the inhomogenous equation $A\mathbf{x} = \mathbf{b}$ from part (b). (FALSE). You can check $A\mathbf{x} \neq \mathbf{0}$.

$\begin{bmatrix} 1 \\ 7 \\ -4 \\ 3 \end{bmatrix}$ is a solution to the inhomogenous equation $A\mathbf{x} = \mathbf{b}$ from part (b). (TRUE). You can check $A\mathbf{x} = \mathbf{0}$.

$\begin{bmatrix} -1 \\ 12 \\ -4 \\ 3 \end{bmatrix}$ is a solution to the inhomogenous equation $A\mathbf{x} = \mathbf{b}$ from part (a). (TRUE). You can check $A\mathbf{x} = \mathbf{b}$.