MSE Hand in 6 - Solution

Assignment 1

Information is transmitted through a fiber optic cable in the form of bits, which can have the value of 0 or 1 . However, there is noise on the line, so a transmitted 0 will, with a probability of 1/10, be changed to 1 during transmission. A transmitted 1 will, with a probability of 1/5, be changed to 0 during transmission. The probability that a randomly chosen bit is transmitted with the value 0 is 2/3.

a. A bit with the value 0 is received. What is the probability that the received bit was transmitted with the value 0?

Exercise 1 A_0: Sent as 0 A_1: Sent as 1 B_0: Received as 0 B_1: Received as 1 Given: Independent events

$$P(A_0) = \frac{2}{3}, \quad P(A_1) = \frac{1}{3},$$
$$P(B_1 \mid A_0) = \frac{1}{10}, \quad P(B_0 \mid A_0) = \frac{9}{10},$$
$$P(B_0 \mid A_1) = \frac{1}{5}, \quad P(B_1 \mid A_1) = \frac{4}{5}.$$

Find $P(A_0 \mid B_0)$.

Using Bayes' theorem:

$$P(A_0 \mid B_0) = \frac{P(B_0 \mid A_0)\, P(A_0)}{P(B_0)}$$

The law of total probability:

$$\begin{aligned} P(B_0) &= P(B_0 \mid A_0) \cdot P(A_0) + P(B_0 \mid A_1) \cdot P(A_1) \\ &= \frac{9}{10} \cdot \frac{2}{3} + \frac{1}{5} \cdot \frac{1}{3} \\ &= \frac{18}{30} + \frac{1}{15} \\ &= \frac{9}{15} + \frac{2}{30} = \frac{20}{30} = \frac{2}{3}. \end{aligned}$$

$$P(A_0 \mid B_0) = \frac{\frac{9}{10} \cdot \frac{2}{3}}{\frac{2}{3}} = \underline{\underline{\frac{9}{10}}}$$

b. A bit with the value 1 is received. What is the probability that the received bit was transmitted with the value 1?

Find $P(A_1 \mid B_1)$

$$\text{Bayes } P(A_1 \mid B_1) = \frac{P(B_1 \mid A_1) \cdot P(A_1)}{P(B_1)}$$

Law of total probability

$$
\begin{aligned}
P(B_1) &= P(B_1 \mid A_0) P(A_0) + P(B_1 \mid A_1) \cdot P(A_1) \\
&= \frac{1}{10} \cdot \frac{2}{3} + \frac{4}{5} \cdot \frac{1}{3} \\
&= \frac{2}{30} + \frac{4}{15} \\
&= \frac{1}{15} + \frac{4}{15} \\
&= \frac{5}{15} \\
&= \frac{1}{3}
\end{aligned}
$$

$$
P(A_1 \mid B_1) = \frac{\frac{4}{5} \cdot \frac{1}{3}}{\frac{1}{3}} = \underline{\underline{\frac{4}{5}}}
$$

c. What is the probability that the received bit will be different from the transmitted bit?

$$
\begin{aligned}
P(\text{diff}) &= P(B_1 \mid A_0) \cdot P(A_0) + P(B_0 \mid A_1) \cdot P(A_1) \\
&= \cdot \frac{1}{10} \cdot \frac{2}{3} + \frac{1}{5} \cdot \frac{1}{3} \\
&= \frac{2}{30} + \frac{1}{15} = \frac{1}{15} + \frac{1}{15} = \underline{\underline{\frac{2}{15}}}
\end{aligned}
$$

Assignment 2

a. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Row echelon form

$\begin{bmatrix} 0 & -4 & 1 \\ 2 & 0 & 0 \\ 1 & -3 & 3 \end{bmatrix}$ No form

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & -7 \end{bmatrix}$ Row of zeros so no form

$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ No form

b. $\left[\begin{array}{ccc|c} -3 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ x_3 is free

$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right]$ x_2, x_4 are free

$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ no free variables

c. Mark each as true or false.

(a) The number of pivot columns in a reduced matrix is the same as the number of free variables.
False.

- The number of pivot columns corresponds to the number of **basic variables** in the system.
- The number of free variables is determined by subtracting the number of pivot columns from the total number of variables.

- In general:

$$\text{Number of variables} = \text{Number of pivot columns} + \text{Number of free variables.}$$

(b) If a system is consistent, there will be at least one free variable.
False.

- A consistent system guarantees at least one solution, but it does not necessarily imply the presence of free variables.
- A consistent system can have:
  - No free variables (unique solution, where all variables are basic), or
  - At least one free variable (infinitely many solutions).
- Example:
  - A unique solution for $x + y = 2$ has no free variables.
  - An underdetermined system like $x + y = 2,\ 0 = 0$ can have a free variable.

(c) The number of pivot columns determines the number of basic variables.
True.

- Pivot columns correspond to the columns in the matrix associated with the basic variables in the system.
- Each pivot column indicates a basic variable, while non-pivot columns correspond to free variables.

(d) If two matrices are row equivalent, they represent two systems of linear equations with the same set of solutions. **True.**

- Row equivalence means one matrix can be obtained from the other via elementary row operations, which do not alter the solution set of the system.
- Hence, two row-equivalent matrices represent systems with the same solutions.

(e) It is possible for a system of linear equations to have exactly two solutions.
False.

- A system of linear equations can have:
  - **No solutions** (inconsistent),
  - **Exactly one solution** (consistent, no free variables), or
  - **Infinitely many solutions** (consistent, with at least one free variable).
- The structure of linear equations does not allow for exactly two solutions because the solution set is either a point, a line, or an empty set, depending on the consistency and number of free variables.

Assignment 3

a. Solve each of the following systems by writing down the augmented matrix and finding the reduced row echelon form.

$$\begin{aligned} 5x_1 - 5x_2 + 5x_3 &= 5 \\ 2x_1 + 4x_2 - 6x_3 &= 12 \\ 10x_1 - 5x_2 + 5x_3 &= 30 \end{aligned}$$

Augmented

$$\begin{bmatrix} 5 & -5 & 5 & 5 \\ 2 & 4 & -6 & 12 \\ 10 & -5 & 5 & 30 \end{bmatrix} \overset{r_1 \to \frac{1}{5}r_1}{\underset{r_3 \to \frac{1}{5}r_3}{r_2 \to \frac{1}{2}r_2 1}} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 2 & -3 & 6 \\ 2 & -1 & 1 & 6 \end{bmatrix}$$

$$r_2 \to r_2 - r_1 \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 3 & -4 & 5 \\ 2 & -1 & 1 & 6 \end{bmatrix} r_3 \to r_3 - 2r_1 \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 1 & -1 & 4 \end{bmatrix}$$

$$r_2 \longleftrightarrow r_3 \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 4 \\ 0 & 3 & -4 & 5 \end{bmatrix} r_3 \to r_3 - 3r_2 \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -1 & -7 \end{bmatrix}$$

$$\overset{r_3 \to -1 \cdot r_3}{\sim} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 7 \end{bmatrix} \overset{\substack{r_2 \to r_2 + r_3 \\ r_1 \to r_1 - r_3}}{} \begin{bmatrix} 1 & -1 & 0 & -6 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\overset{r_1 \to r_1 + r_2}{} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$x_1 = 5$
$x_2 = 11$
$x_3 = 7$

b.

$$x_1 + 2x_2 + 3x_3 = 0$$
$$4x_1 + 5x_2 + 6x_3 = 0$$

Augmented

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \end{bmatrix} \overset{r_2 \to r_2 - 4r_1}{} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix}$$

$$\overset{r_2 \to \frac{-1}{3} r_2}{} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \overset{r_1 \to r_1 - 2r_2}{} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned} \Leftrightarrow \begin{aligned} x_1 &= x_3 \\ x_2 &= -2x_3 \\ x_3 &= x_3 \end{aligned}$$

Assignment 4

a. Is the system consistent?

$$x_1 + 2x_3 + 4x_4 = 6$$
$$4x_2 - 6x_3 - 3x_4 = 0$$
$$4x_1 + 8x_2 - 4x_3 + 10x_4 = 1$$

Augmented matrix:

$$\begin{bmatrix} 1 & 0 & 2 & 4 & 6 \\ 0 & 4 & -6 & -3 & 0 \\ 4 & 8 & -4 & 10 & 1 \end{bmatrix} \overset{r_3 \to r_3 - 4r_1}{} \begin{bmatrix} 1 & 0 & 2 & 4 & -6 \\ 0 & 4 & -6 & -3 & 0 \\ 0 & 8 & -12 & -6 & -23 \end{bmatrix}$$

$$\overset{r_3 \to r_3 - 2r_2}{} \begin{bmatrix} 1 & 0 & 2 & 4 & 6 \\ 0 & 4 & -6 & -3 & 0 \\ 0 & 0 & 0 & 0 & -23 \end{bmatrix}$$

since we get $0 = -23$, the system is inconsistent.

b. Find h such that the system is consistent.

$$\begin{cases} x_1 - 2x_2 + 4x_3 = 1 \\ 2x_2 + x_3 = -5 \\ 2x_1 + 10x_3 = h \end{cases}$$

Augmented matrix:

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 2 & 1 & -5 \\ 2 & 0 & 10 & h \end{bmatrix} \overset{r_3 \to r_3 - 2r_1}{} \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 2 & 1 & -5 \\ 0 & 4 & 2 & h-2 \end{bmatrix}$$

$$r_3 \to r_3 - 2r_2 \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & 0 & h+8 \end{bmatrix}$$

The system is consistent if $h + 8 = 0 \Rightarrow h = -8$.

Assignment 5

Given three points $(1, 2), (4, 5)$ and $(6, 4)$ find a second degree polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ that passes through all three points. That is, find a_0, a_1 and a_2 to fulfill the following equations

$$a_0 + a_1 \cdot 1 + a_2 \cdot 1^2 = 2$$
$$a_0 + a_1 \cdot 4 + a_2 \cdot 4^2 = 5$$
$$a_0 + a_1 \cdot 6 + a_2 \cdot 6^2 = 4$$

Just solve the system of equations.

$$a_0 + a_1 \cdot 1 + a_2 \cdot 1^2 = 2 \Leftrightarrow a_0 + a_1 + a_2 = 2$$
$$a_0 + a_1 \cdot 4 + a_2 \cdot 4^2 = 5 \Leftrightarrow a_0 + 4a_1 + 16a_2 = 5$$
$$a_0 + a_1 \cdot 6 + a_2 \cdot 6^2 = 4 \Leftrightarrow a_0 + 6a_1 + 36a_2 = 4$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 4 & 16 & 5 \\ 1 & 6 & 36 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{5} \\ 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & -\frac{3}{10} \end{bmatrix}$$

$$\underline{\underline{a_0 = -\frac{1}{5}, \quad a_1 = \frac{5}{2}, \quad a_2 = -\frac{3}{10}}}$$

Assignment 6

Axel is twice as old as Bob. Caroline is 5 years older than Bob. The combined age of Axel, Caroline, and Danny is 110 years. Also, the sum of Axel and Bob's ages is the same as the sum of Caroline and Danny's ages.

a. What is the system of equations that describes the problem?

$$\begin{cases} A - 2B = 0 \\ B - C = -5 \\ A + C + D = 110 \\ A + B - C - D = 0 \end{cases}$$

b. Solve the system of equations.

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -5 \\ 1 & 0 & 1 & 1 & 110 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 44 \\ 0 & 1 & 0 & 0 & 22 \\ 0 & 0 & 1 & 0 & 27 \\ 0 & 0 & 0 & 1 & 39 \end{bmatrix}$$