MSE Hand in 1 - solution

The assignments are to be solved in pairs. Each pair can only hand in one solution.
Submission by the end of class.
The assignments must be solved by hand.

Assignment 1

Rearrange each of the formulas below such that x becomes the subject

(a) $4 = e^x + y$

$$
\begin{aligned}
4 &= e^x + y \\
4 - y &= e^x && \text{(Subtract } y \text{ from both sides)} \\
e^x &= 4 - y \\
\ln(e^x) &= \ln(4 - y) && \text{(Take the natural logarithm of both sides)} \\
x &= \ln(4 - y) && \text{(Since } \ln(e^x) = x\text{)}
\end{aligned}
$$

Final rearranged formula:

$$x = \ln(4 - y)$$

(b) $y = 10^{z+1/x}$

$$
\begin{aligned}
y &= 10^{z+\frac{1}{x}} \\
\log_{10} y &= z + \frac{1}{x} && \text{(Take the logarithm base 10 of both sides)} \\
\log_{10} y - z &= \frac{1}{x} && \text{(Subtract } z \text{ from both sides)} \\
\frac{1}{\log_{10} y - z} &= x && \text{(Take the reciprocal of both sides)} \\
x &= \frac{1}{\log_{10} y - z} && \text{(Simplify)}
\end{aligned}
$$

Final rearranged formula:

$$x = \frac{1}{\log_{10} y - z}$$

(c) $10 = \ln{(5x)}^2$

$$
\begin{aligned}
10 &= \ln{(5x)}^2 \\
10 &= 2\ln|5x| && \text{(Using the identity } \ln a^b = b\ln|a|\text{)} \\
5 &= \ln|5x| && \text{(Divide both sides by 2)} \\
e^5 &= |5x| && \text{(Exponentiate both sides)} \\
5x &= \pm e^5 && \text{(Remove the absolute value, resulting in } \pm\text{)} \\
x &= \pm\frac{e^5}{5} && \text{(Divide both sides by 5)}
\end{aligned}
$$

Final rearranged formula:

$$x = \pm\frac{e^5}{5}$$

(d) $k + 1 = \log_2\left(\frac{n \cdot x}{2}\right)$

$$\begin{aligned}
k + 1 &= \log_2\left(\frac{nx}{2}\right) \\
2^{k+1} &= \frac{nx}{2} && \text{(Exponentiate both sides with base 2)} \\
2 \cdot 2^{k+1} &= nx && \text{(Multiply both sides by 2)} \\
2^{k+2} &= nx && \text{(Simplify the left side)} \\
x &= \frac{2^{k+2}}{n} && \text{(Divide both sides by } n\text{)}
\end{aligned}$$

Final rearranged formula:

$$x = \frac{2^{k+2}}{n}$$

Assignment 2

The error associated with statistical uncertainty is given by

$$E = Z_{1-\alpha}\frac{\sigma}{\sqrt{n}}$$

where $Z_{1-\alpha}$ is the z-score associated with the standard normal distribution, σ is the standard deviation and n is the sample size.

(a) Rearrange the above formula such that sample size becomes the subject

$$\begin{aligned}
E &= Z_{1-\alpha}\frac{\sigma}{\sqrt{n}} \\
E\sqrt{n} &= Z_{1-\alpha}\sigma && \text{(Multiply both sides by } \sqrt{n}\text{)} \\
\sqrt{n} &= \frac{Z_{1-\alpha}\sigma}{E} && \text{(Divide both sides by } E\text{)} \\
n &= \left(\frac{Z_{1-\alpha}\sigma}{E}\right)^2 && \text{(Square both sides)}
\end{aligned}$$

Final rearranged formula:

$$n = \left(\frac{Z_{1-\alpha}\sigma}{E}\right)^2$$

(b) Assuming a z-score of 1.96, and standard deviation of 2, what sample size is needed in order to have an error of no more than 0.5?

To find the required sample size n such that the error E is no more than 0.5, given a z-score of 1.96 and a standard deviation $\sigma = 2$, we use the formula:

$$n = \left(\frac{Z_{1-\alpha}\sigma}{E}\right)^2$$

Given:

- $Z_{1-\alpha} = 1.96$
- $\sigma = 2$
- $E = 0.5$

Calculations:

$$\begin{aligned} n &= \left(\frac{Z_{1-\alpha}\sigma}{E}\right)^2 \\ &= \left(\frac{1.96 \times 2}{0.5}\right)^2 \\ &= \left(\frac{3.92}{0.5}\right)^2 \\ &= (7.84)^2 \\ &= 61.4656 \end{aligned}$$

Since the sample size n must be a whole number, and to ensure the error does not exceed 0.5, we round up to the next whole number.

Final Answer:

$$\boxed{n \geq 62}$$

Assignment 3

Find the greatest common divisor and the least common multiple of the pairs of integers below

(a) $2^2 \cdot 3^3 \cdot 5^5$ and $2^5 \cdot 3^3 \cdot 5^2$

Greatest Common Divisor (gcd):

For each common prime, take the **minimum exponent**:

$$\gcd(a, b) = 2^{\min(2,5)} \cdot 3^{\min(3,3)} \cdot 5^{\min(5,2)} = 2^2 \cdot 3^3 \cdot 5^2$$

Compute the gcd:

$$\begin{aligned} \gcd(a, b) &= 2^2 \cdot 3^3 \cdot 5^2 \\ &= 4 \cdot 27 \cdot 25 \\ &= 2700 \end{aligned}$$

Least Common Multiple (lcm):

For each prime, take the **maximum exponent**:

$$\operatorname{lcm}(a, b) = 2^{\max(2,5)} \cdot 3^{\max(3,3)} \cdot 5^{\max(5,2)} = 2^5 \cdot 3^3 \cdot 5^5$$

Compute the lcm:

$$\begin{aligned} \operatorname{lcm}(a, b) &= 2^5 \cdot 3^3 \cdot 5^5 \\ &= 32 \cdot 27 \cdot 3125 \\ &= 2,700,000 \end{aligned}$$

Final Answers:

$$\boxed{\gcd(a, b) = 2700}$$

$$\boxed{\operatorname{lcm}(a, b) = 2,700,000}$$

(b) $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ and $2^{11} \cdot 3^9 \cdot 11 \cdot 17^{14}$

Greatest Common Divisor (gcd):

For each common prime, take the **minimum exponent**:

$$\gcd(a,b) = 2^{\min(1,11)} \cdot 3^{\min(1,9)} \cdot 11^{\min(1,1)} = 2^1 \cdot 3^1 \cdot 11^1$$

Compute the gcd:

$$\begin{aligned}\gcd(a,b) &= 2^1 \cdot 3^1 \cdot 11^1 \\ &= 2 \cdot 3 \cdot 11 \\ &= 66\end{aligned}$$

Least Common Multiple (lcm):

For each prime, take the **maximum exponent**:

$$\text{lcm}(a,b) = 2^{\max(1,11)} \cdot 3^{\max(1,9)} \cdot 5^{\max(1,0)} \cdot 7^{\max(1,0)} \cdot 11^{\max(1,1)} \cdot 13^{\max(1,0)} \cdot 17^{\max(0,14)} = 2^{11} \cdot 3^9 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^{14}$$

Compute the lcm:

$$\begin{aligned}\text{lcm}(a,b) &= 2^{11} \cdot 3^9 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^{14} \\ &= 2048 \times 19683 \times 5 \times 7 \times 11 \times 13 \times 17^{14} \\ &= 201,755,473,920 \times 17^{14} \\ &\approx 201,755,473,920 \times 1.683 \times 10^{17} \\ &= 3.39794854893536 \times 10^{23}\end{aligned}$$

Final Answers:

$$\gcd(a,b) = 66$$

$$\text{lcm}(a,b) \approx 3.398 \times 10^{23}$$

For two positive integers a and b the product can be calculated as $a \cdot b = \gcd(a,b) \cdot \text{lcm}(a,b)$

(c) Verify that the rule holds for the integers in (a) and (b).

For two positive integers a and b, the product can be calculated as:

$$a \cdot b = \gcd(a,b) \cdot \text{lcm}(a,b)$$

First Pair of Integers:

Given:

$$a = 2^2 \cdot 3^3 \cdot 5^5$$

$$b = 2^5 \cdot 3^3 \cdot 5^2$$

Previously calculated:

$$\gcd(a,b) = 2^2 \cdot 3^3 \cdot 5^2$$

$$\text{lcm}(a,b) = 2^5 \cdot 3^3 \cdot 5^5$$

Compute $a \cdot b$:

$$\begin{aligned}a \cdot b &= \left(2^2 \cdot 3^3 \cdot 5^5\right) \cdot \left(2^5 \cdot 3^3 \cdot 5^2\right) \\ &= 2^{2+5} \cdot 3^{3+3} \cdot 5^{5+2} \\ &= 2^7 \cdot 3^6 \cdot 5^7\end{aligned}$$

Compute $\gcd(a,b)\cdot\text{lcm}(a,b)$:

$$\begin{aligned}\gcd(a,b)\cdot\text{lcm}(a,b) &= \left(2^2\cdot 3^3\cdot 5^2\right)\cdot\left(2^5\cdot 3^3\cdot 5^5\right)\\ &= 2^{2+5}\cdot 3^{3+3}\cdot 5^{2+5}\\ &= 2^7\cdot 3^6\cdot 5^7\end{aligned}$$

Conclusion:

$$a\cdot b=\gcd(a,b)\cdot\text{lcm}(a,b)$$

Second Pair of Integers:

Given:

$$a=2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13=2^1\cdot 3^1\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1$$

$$b=2^{11}\cdot 3^9\cdot 11\cdot 17^{14}=2^{11}\cdot 3^9\cdot 11^1\cdot 17^{14}$$

Previously calculated:

$$\gcd(a,b)=2^1\cdot 3^1\cdot 11^1$$

$$\text{lcm}(a,b)=2^{11}\cdot 3^9\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^{14}$$

Compute $a\cdot b$:

$$\begin{aligned}a\cdot b &= \left(2^1\cdot 3^1\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1\right)\cdot\left(2^{11}\cdot 3^9\cdot 11^1\cdot 17^{14}\right)\\ &= 2^{1+11}\cdot 3^{1+9}\cdot 5^1\cdot 7^1\cdot 11^{1+1}\cdot 13^1\cdot 17^{14}\\ &= 2^{12}\cdot 3^{10}\cdot 5^1\cdot 7^1\cdot 11^2\cdot 13^1\cdot 17^{14}\end{aligned}$$

Compute $\gcd(a,b)\cdot\text{lcm}(a,b)$:

$$\begin{aligned}\gcd(a,b)\cdot\text{lcm}(a,b) &= \left(2^1\cdot 3^1\cdot 11^1\right)\cdot\left(2^{11}\cdot 3^9\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^{14}\right)\\ &= 2^{1+11}\cdot 3^{1+9}\cdot 5^1\cdot 7^1\cdot 11^{1+1}\cdot 13^1\cdot 17^{14}\\ &= 2^{12}\cdot 3^{10}\cdot 5^1\cdot 7^1\cdot 11^2\cdot 13^1\cdot 17^{14}\end{aligned}$$

Conclusion:

$$a\cdot b=\gcd(a,b)\cdot\text{lcm}(a,b)$$

(d) If the product of two integers is $2^7 3^8 5^2 7^{11}$ and their greatest common divisor is $2^3 3^4 5$ what is their least common multiple?

Given:

$$a\cdot b=2^7\cdot 3^8\cdot 5^2\cdot 7^{11}$$

$$\gcd(a,b)=2^3\cdot 3^4\cdot 5^1$$

We know that:

$$\text{lcm}(a,b)=\frac{a\cdot b}{\gcd(a,b)}$$

Compute $\text{lcm}(a,b)$:

$$\begin{aligned}\text{lcm}(a,b) &= \frac{2^7\cdot 3^8\cdot 5^2\cdot 7^{11}}{2^3\cdot 3^4\cdot 5^1}\\ &= 2^{7-3}\cdot 3^{8-4}\cdot 5^{2-1}\cdot 7^{11} && \text{(Subtract exponents)}\\ &= 2^4\cdot 3^4\cdot 5^1\cdot 7^{11}\end{aligned}$$

Final Answer:

$$\text{lcm}(a,b)=2^4\cdot 3^4\cdot 5^1\cdot 7^{11}$$

Assignment 4

Find the values

(a) $231 \mod 12$ We have:

$$231 \div 12 = 19 \text{ with a remainder of } 3$$

Therefore:

$$231 \mod 12 = 3$$

(b) $88 \mod 12$ We have:

$$88 \div 12 = 7 \text{ with a remainder of } 4$$

Therefore:

$$88 \mod 12 = 4$$

(c) $599 \mod 9$ We have:

$$599 \div 9 = 66 \text{ with a remainder of } 5$$

Therefore:

$$599 \mod 9 = 5$$

(d) $400 \mod 9$

$$400 \div 9 = 44 \text{ with a remainder of } 4$$

Therefore:

$$400 \mod 9 = 4$$

Use (a)-(d) to do the following

(e) Check if $(231 + 88) \mod 12 = (231 \mod 12 + 88 \mod 12) \mod 12$

$$231 + 88 = 319$$

$$LHS = 319 \mod 12$$

Calculate $319 \mod 12$:

$$319 \div 12 = 26 \text{ with a remainder of } 7$$

So:

$$LHS = 7$$

Compute the right-hand side (RHS):

$$231 \mod 12 = 3, \quad 88 \mod 12 = 4$$

$$RHS = (3 + 4) \mod 12 = 7 \mod 12 = 7$$

Since:

$$LHS = RHS = 7$$

Conclusion:

$$(231 + 88) \mod 12 = (231 \mod 12 + 88 \mod 12) \mod 12$$

(f) Check if $(599 + 400) \mod 9 = (599 \mod 9 + 400 \mod 9) \mod 9$ Compute the left-hand side (LHS):

$$599 + 400 = 999$$

$$LHS = 999 \mod 9$$

Calculate $999 \mod 9$:

$$999 \div 9 = 111 \text{ with a remainder of } 0$$

So:

$$LHS = 0$$

Compute the right-hand side (RHS):

$$599 \mod 9 = 5, \quad 400 \mod 9 = 4$$

$$RHS = (5 + 4) \mod 9 = 9 \mod 9 = 0$$

Since:

$$LHS = RHS = 0$$

Conclusion:

$$(599 + 400) \mod 9 = (599 \mod 9 + 400 \mod 9) \mod 9$$

Therefore, the rule holds for these cases, and actually the rule holds for any sum of integers:

$$(a + b) \mod m = (a \mod m + b \mod m) \mod m$$